3-23 $\rho(\bar{x}) = \begin{cases} Cr & r \le a \\ \emptyset & r > a \end{cases}$; $Q = \int \rho\, d^3x = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} Cr \cdot r^2 dr$

$= C\pi a^4$

a) Find $\vec{E}(\vec{r})$, $V(\vec{r})$: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\bar{x}')\, d^3x'}{|\bar{x} - \bar{x}'|}$

$$V = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi' \int_0^{a} dr' \int_0^{\pi} d\theta' \frac{Cr' \cdot r'^2 \sin\theta'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

Let $u = \cos\theta'$; $du = -\sin\theta' d\theta'$

$$V = \frac{-1}{2\varepsilon_0} \int_0^{a} dr' \int_1^{-1} \frac{Cr'^3\, du}{[r^2 + r'^2 - 2rr'u]^{1/2}} = \frac{-C}{2\varepsilon_0} \int_0^{a} dr'\, r'^3 \frac{[r^2 + r'^2 - 2rr'u]^{+1/2} \cdot 2}{-2rr'}$$

$$V = \frac{+C}{2\varepsilon_0 r} \int_0^{a} r'^2 dr' \left[|r + r'| - |r - r'| \right]$$

$r > a$ $\quad |r + r'| - |r - r'| = 2r'$;

$$V = \int_0^{a} \frac{C}{\varepsilon_0 r} r'^3 dr' = \frac{Ca^4}{4\varepsilon_0 r}$$

$r < a$ $\quad$ i) $r > r'$, $|r + r'| - |r - r'| = 2r'$, $V_i = \int_0^{r} \frac{C}{\varepsilon_0 r} r'^3 dr' = \frac{Cr^3}{4\varepsilon_0}$

ii) $r < r'$; $|r + r'| - |r - r'| = 2r$, $V_{ii} = \int_r^{a} \frac{C}{\varepsilon_0} r'^2 dr' = \frac{C}{\varepsilon_0}\left[\frac{a^3}{3} - \frac{r^3}{3}\right]$

$V = V_i + V_{ii}$ (to get the potential from all charge)

$$V = \frac{C}{\varepsilon_0}\left[\frac{a^3}{3} - \frac{r^3}{3} + \frac{r^3}{4}\right] = \frac{C}{\varepsilon_0}\left[\frac{a^3}{3} - \frac{r^3}{12}\right]$$

$\vec{E} = -\vec{\nabla} V$

$r > a$ $\quad \vec{E} = -\frac{Ca^4}{4\varepsilon_0} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = + Ca^4 / 4\varepsilon_0 r^2$

$r < a$ $\quad \vec{E} = +\frac{C}{12\varepsilon_0} \frac{\partial r^3}{\partial r} = Cr^2 / 4\varepsilon_0$

3-23 b) If we add a surface charge σ_0, the addition to the potential is:

$$V_s = \frac{1}{4\pi\varepsilon_0} \oint \frac{\sigma\, dA}{|\bar{x}-\bar{x}'|}$$

$$= \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma_0 a^2\, d\phi \sin\theta\, d\theta}{[r^2+a^2-2ar\cos\theta]^{1/2}}$$

again let $u=\cos\theta$, $du=-\sin\theta\, d\theta$

$$V_s = -\frac{\sigma_0 a^2}{2\varepsilon_0} \int_{+1}^{-1} \frac{du}{[r^2+a^2-2aru]^{1/2}} = \frac{+\sigma_0 a^2}{2\varepsilon_0} \frac{[r^2+a^2-2aru]^{1/2}}{ar}$$

$$V_s = \frac{\sigma_0 a}{2\varepsilon_0 r}\left[|r+a| - |r-a|\right]$$

$r>a$ for $r>a$, $V_s = \sigma_0 a^2/\varepsilon_0 r$,

$r<a$ $V_s = \sigma_0 a/\varepsilon_0$

$r>a$ $V = \frac{Ca^4}{4\varepsilon_0 r} + \frac{\sigma_0 a^2}{r}$, $\vec{E} = -\bar{\nabla}V = \frac{Ca^4}{4\varepsilon_0 r^2}\hat{r} + \frac{\sigma_0 a^2}{\varepsilon_0 r^2}\hat{r}$

$r<a$ $V = \frac{C}{\varepsilon_0}\left[\frac{a^3}{3} - \frac{r^3}{12}\right] + \frac{\sigma_0 a}{\varepsilon_0}$, $\vec{E} = -\bar{\nabla}V = \frac{Cr^2}{4\varepsilon_0}\hat{r}$

$$E_{2n} - E_{1n} = \sigma_0 a^2/\varepsilon_0 r^2 \Big|_{r=a} = \frac{\sigma_0}{\varepsilon_0},$$ as expected from boundary conditions