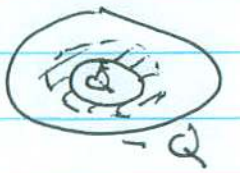
6-16 Spherical capacitor has $C = 4\pi\varepsilon_0 \frac{ab}{b-a}$

a) Fill half way w/ dielectric



$$\oint \vec{D}\cdot d\vec{A} = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

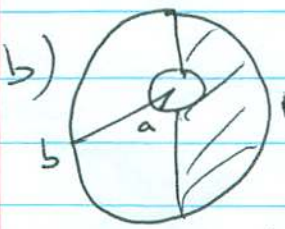
$$V = \int_+^- \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi}\left[\int_a^{\frac{a+b}{2}} \frac{dr}{\epsilon r^2} + \int_{\frac{a+b}{2}}^{b} \frac{dr}{\varepsilon_0 r^2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\int_a^{\frac{a+b}{2}} \frac{dr}{\kappa r^2} + \int_{\frac{a+b}{2}}^{b} \frac{dr}{r^2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{-1}{\kappa r}\Big|_a^{\frac{a+b}{2}} + \frac{-1}{r}\Big|_{\frac{a+b}{2}}^{b}\right]$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\kappa a} - \frac{2}{\kappa(a+b)} + \frac{2}{a+b} - \frac{1}{b}\right] = \frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{\kappa ab} + \frac{2}{a+b}\frac{\kappa-1}{\kappa}\right)$$

$$C = Q/V = 4\pi\varepsilon_0\left[\frac{b-a}{\kappa ab} + \frac{2}{a+b}\frac{\kappa-1}{\kappa}\right]^{-1} = \frac{4\pi\varepsilon_0 \kappa ab(a+b)}{(\kappa a+b)(b-a)}$$

b)



$\vec{E} = E_r\hat{r}$ but $\vec{E}_t$ must be continuous $\therefore$ E_r is continuous

In vacuum half ($\because$ dielectric half)

$E_r = \frac{C_0}{r^2}$;

$$V = \int E_r\, dr = V(b) - V(a) = -C_0\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$= C_0\left(\frac{1}{a} - \frac{1}{b}\right) = C_0\frac{b-a}{ab}$$

if $V = V(b) - V(a)$, $C_0 = V\cdot\frac{ab}{b-a}$

at $r=a$, $Q = \oint D_r\, dA = \frac{\varepsilon_0 C_0}{a^2}\cdot\frac{1}{2}(4\pi a^2) + \frac{\varepsilon C_0}{a^2}\cdot\frac{1}{2}(4\pi a^2)$

$$= \varepsilon_0 C(1+\kappa)2\pi$$

$$C = \frac{Q}{V} = \frac{\varepsilon_0 V\frac{ab}{(b-a)}(1+\kappa)2\pi}{V} = \frac{2\pi\varepsilon_0(\kappa+1)ab}{b-a}$$